

CHAPTER 14

Nonlinear Dispersion and the Variational Method

The nonlinear effects found in the study of water waves are typical of dispersive systems in general. Periodic wavetrains, similar to those of Stokes and Korteweg-deVries, are found in most systems and these are the basic solutions corresponding to the elementary solutions $ae^{ikx-i\omega t}$ of linear theory. In nonlinear theory, the solutions are no longer sinusoidal, but the existence of periodic solutions in $\theta = kx - \omega t$ can be shown explicitly in the simpler cases and inferred from the Stokes expansion in others. The main nonlinear effect is not the difference in functional form, it is the appearance of amplitude dependence in the dispersion relation. This leads to new qualitative behavior, not merely to the correction of linear formulas. Superposition of solutions is not available to generate more general wavetrains, but modulation theory can be studied directly. The theory can be developed in general using the variational approach of Section 11.7. The formulation will be studied in detail in this chapter and the justification as a formal perturbation method will be given to complete the earlier discussion. Detailed applications of the theory are then given in Chapters 15 and 16.

Another specific consequence of nonlinearity is the existence of solitary waves. Waves with these profiles would disperse in the linear theory, but the nonlinearity counterbalances the dispersion to produce waves of permanent shape. Solitary waves are found, in the first instance, as limiting cases of the periodic wavetrains, but recent work on their interactions and their production from arbitrary initial data has shown that their special structure is of separate importance. We shall return to these topics in Chapter 17.

For waves of moderately small amplitude in what might be called "near-linear theory," further results may be obtained by perturbation methods based on small amplitude expansions. In particular we may return to the Fourier analysis description and study the small nonlinear interactions of the Fourier components. The interactions transfer energy between different components and, through product terms in the equations,

generate new components from existing ones. These interactions can be followed effectively when only a few components are involved. We shall include typical results as appropriate, but the main emphasis is on methods that extend to the fully nonlinear case. From a Fourier analysis viewpoint, the nonlinear wavetrains and solitary waves are already quite complicated distributions of Fourier components with involved interactions maintaining a balance. The developments emphasized here build directly on these special structures without attempting to disentangle them into their components. However, in the near-linear case there are interesting and informative relations between the two points of view.

14.1 A Nonlinear Klein-Gordon Equation

It is useful to have a simple example to motivate and illustrate the steps in the development of the general theory. For this purpose a nonlinear version of the Klein-Gordon equation is particularly useful and is even simpler than the Korteweg-deVries equation, which would be the other obvious choice. We take the equation

$$\varphi_{tt} - \varphi_{xx} + V'(\varphi) = 0, \quad (14.1)$$

where $V'(\varphi)$ is some reasonable nonlinear function of φ which is chosen as the derivative of a potential energy $V(\varphi)$ for later convenience. Equation 14.1 is not only a useful model; it arises in a variety of physical situations. This is especially true of the case $V'(\varphi) = \sin \varphi$, which almost inevitably has become known as the Sine-Gordon equation! An account of the physical problems in which this form arises is given by Barone et al. (1971), following a briefer version by Scott (1970, p. 250). Its first appearance is not in wave problems at all, but in the study of the geometry of surfaces with Gaussian curvature $K = -1$. In fact some of the transformation methods developed there have been remarkably valuable in finding solutions for interacting solitary waves, as will be discussed in Chapter 17. More recent problems listed by the same authors include:

1. Josephson junction transmission lines, where $\sin \varphi$ is the Josephson current across an insulator between two superconductors, the voltage being proportional to φ_t .
2. Dislocations in crystals, where the occurrence of $\sin \varphi$ is due to the periodic structure of rows of atoms.
3. The propagation in ferromagnetic materials of waves carrying rotations of the direction of magnetization.

4. Laser pulses in two state media, where the variables can also be described in terms of a rotating vector.

Scott further describes his construction of a mechanical model with rigid pendula attached at close intervals along a stretched wire. Torsional waves propagating down the wire obey the wave equation and the pendula supply a restoring force proportional to $\sin \varphi$, where φ is the angular displacement. Scott was able to generate the waves corresponding to many of the solutions of the Sine-Gordon equation.

Equation 14.1 has also been discussed by Schiff (1951), with a cubic nonlinearity, and by Perring and Skyrme (1962), with the $\sin \varphi$ term, in tentative investigations of elementary particles.

In this chapter the analysis applies for general $V(\varphi)$ with appropriate properties. The choice

$$V(\varphi) = \frac{1}{2}\varphi^2 + \sigma\varphi^4$$

is both the simplest to bear in mind and the correct expansion in the near-linear theory for even functions $V(\varphi)$. The small amplitude expansion of the Sine-Gordon equation has $\sigma = -1/24$.

We first check the existence of periodic wavetrains. They are obtained as usual by taking

$$\varphi = \Psi(\theta), \quad \theta = kx - \omega t. \quad (14.2)$$

On substitution we have

$$(\omega^2 - k^2)\Psi_{\theta\theta} + V'(\Psi) = 0 \quad (14.3)$$

and the immediate integral

$$\frac{1}{2}(\omega^2 - k^2)\Psi_{\theta}^2 + V(\Psi) = A. \quad (14.4)$$

We use A for the constant of integration, although earlier it was used to denote the complex amplitude in linear problems. Only the real amplitude a will appear in the same context, so there should be no confusion. Here A is still an amplitude parameter; in the linear case, $V(\Psi) = \frac{1}{2}\Psi^2$, it is related to the actual amplitude a by $A = \frac{1}{2}a^2$.

The solution of (14.4) may be written

$$\theta = \left\{ \frac{1}{2}(\omega^2 - k^2) \right\}^{1/2} \int \frac{d\Psi}{\{A - V(\Psi)\}^{1/2}}, \quad (14.5)$$

and in the special cases where $V(\Psi)$ is either a cubic, a quartic, or trigonometric, $\Psi(\theta)$ can be expressed in terms of standard elliptic functions. Periodic solutions are obtained when Ψ oscillates between two simple zeros of $A - V(\Psi)$. At the zeros $\Psi_\theta = 0$, and the solution curve has a maximum or a minimum; these points occur at finite values of θ , since (14.5) is convergent when the zeros are simple. If the zeros are denoted by Ψ_1 and Ψ_2 , we shall take the case

$$\Psi_1 \leq \Psi \leq \Psi_2, \quad A - V(\Psi) \geq 0, \quad \omega^2 - k^2 > 0 \quad (14.6)$$

for the present. The period in θ can be normalized to 2π (which is convenient in the linear limit) and we then have

$$2\pi = \left\{ \frac{1}{2}(\omega^2 - k^2) \right\}^{1/2} \oint \frac{d\Psi}{\{A - V(\Psi)\}^{1/2}}, \quad (14.7)$$

where \oint denotes the integral over a complete oscillation of Ψ from Ψ_1 up to Ψ_2 and back. The sign of the square root has to be changed appropriately in the two parts of the cycle. The integral may also be interpreted as a loop integral around a cut from Ψ_1 to Ψ_2 in the complex Ψ plane.

In the linear case $V(\Psi) = \frac{1}{2}\Psi^2$, the periodic solution is

$$\Psi = a \cos \theta, \quad A = \frac{a^2}{2}; \quad (14.8)$$

the amplitude a cancels out in (14.7), which becomes simply the linear dispersion relation

$$\omega^2 - k^2 = 1. \quad (14.9)$$

In the nonlinear case, the amplitude parameter A does not drop out of (14.7) and we have the typical dependence of the dispersion relation on amplitude.

If the amplitude is small and V has the expansion

$$V = \frac{1}{2}\varphi^2 + \sigma\varphi^4 + \dots, \quad (14.10)$$

we have

$$\Psi = a \cos \theta + \frac{1}{8}\sigma a^3 \cos 3\theta + \dots, \quad (14.11)$$

$$\omega^2 - k^2 = 1 + 3\sigma a^2 + \dots, \quad (14.12)$$

$$A = \frac{1}{2}a^2 + \frac{9}{8}\sigma a^4 + \dots \quad (14.13)$$

These are the Stokes expansions, which may be obtained either by direct substitution in (14.3)–(14.4) or by expansion of the exact expressions (14.5) and (14.7) obtained above. It should be noted that a is the amplitude of the first term in (14.11); it differs slightly from the exact amplitude

$$a + \frac{1}{8}\sigma a^3 + \dots$$

14.2 A First Look at Modulations

In the basic case of one dimensional waves in a uniform medium, we saw in Chapter 11 that modulations on a *linear* wavetrain can be described by the equations

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0, \quad (14.14)$$

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x} (C_0 a^2) = 0, \quad (14.15)$$

where $\omega = \omega_0(k)$ is given by the linear dispersion relation and $C_0 = \omega'_0(k)$ is the linear group velocity. (A subscript zero is added now to indicate the linear values.) The crucial qualitative change of nonlinearity is the dependence of ω on a , which couples (14.14) to (14.15). For moderately small amplitudes, ω may be expressed in Stokes fashion as

$$\omega = \omega_0(k) + \omega_2(k)a^2 + \dots, \quad (14.16)$$

and (14.14) becomes

$$\frac{\partial k}{\partial t} + \{ \omega'_0(k) + \omega'_2(k)a^2 \} \frac{\partial k}{\partial x} + \omega_2(k) \frac{\partial a^2}{\partial x} = 0. \quad (14.17)$$

The important coupling term is $\omega_2(k)\partial a^2/\partial x$ because it introduces a term in the derivative of a ; it leads to a correction $O(a)$ to the characteristic velocities. The other new term merely corrects the coefficient of the existing term in $\partial k/\partial x$ and consequently contributes only at the $O(a^2)$ level. Similarly, for small amplitudes, the nonlinear corrections to (14.15) would be various terms of order a^4 which would provide corrections of relative order a^2 to the coefficients of the existing terms in $\partial a^2/\partial x$ and $\partial k/\partial x$. Therefore in the first assessment of nonlinear effects we can

proceed very simply, using only the new dispersion relation, and take

$$\frac{\partial k}{\partial t} + \omega'_0(k) \frac{\partial k}{\partial x} + \omega_2(k) \frac{\partial a^2}{\partial x} = 0, \quad (14.18)$$

$$\frac{\partial a^2}{\partial t} + \omega'_0(k) \frac{\partial a^2}{\partial x} + \omega''_0(k) a^2 \frac{\partial k}{\partial x} = 0. \quad (14.19)$$

By the standard procedure of Chapter 5, the characteristic form of these coupled equations is found to be

$$\frac{1}{2} \left\{ \frac{\omega''_0(k)}{\omega_2(k)} \right\}^{1/2} dk \pm da = 0 \quad (14.20)$$

on characteristics

$$\frac{dx}{dt} = \omega'_0(k) \pm \{ \omega_2(k) \omega''_0(k) \}^{1/2} a. \quad (14.21)$$

It may be verified that additional terms of relative order a^2 added to (14.18)–(14.19) contribute terms only of order a^2 to (14.20)–(14.21).

This simple formulation already shows some remarkable results. In the case $\omega_2 \omega''_0 > 0$, the characteristics are real and the system is hyperbolic. The double characteristic velocity splits under the nonlinear correction and we have the *two* velocities given by (14.21). In general, an initial disturbance or modulating source will introduce disturbances on both families of characteristics. If the disturbance is initially finite in extent, for example a bulge on an otherwise uniform wavetrain, it will eventually split into two. This is completely different from the linear behavior where such a bulge may distort due to the dependence of $C_0(k)$ on k but would not split.

A second consequence of nonlinearity in the hyperbolic case is that “compressive” modulations will distort and steepen in the typical hyperbolic fashion discussed in Part I. This raises the question of multivalued solutions and breaking.

When $\omega_2 \omega''_0 < 0$, the characteristics are imaginary and the system is elliptic. This leads to ill-posed problems in the wave propagation context. Among other things, it means that small perturbations will grow in time and in this sense the periodic wavetrain is *unstable*. The elliptic case turns out to be not uncommon and the modulation theory provides an interesting approach to some aspects of stability theory.

We might note that for Stokes waves in deep water, the dispersion relation (13.124) gives

$$\omega_0(k) = g^{1/2} k^{1/2}, \quad \omega_2(k) = \frac{1}{2} g^{1/2} k^{3/2}, \quad (14.22)$$

so this is an unstable case with $\omega_0''\omega_2 < 0$. This is surprising in view of the long history of the problem and the sometimes controversial arguments about the existence of periodic solutions; throughout these discussions the instability went unrecognized. For the Klein-Gordon example (14.12), we have

$$\omega_0(k) = (k^2 + 1)^{1/2}, \quad \omega_2(k) = \frac{3}{2}\sigma(k^2 + 1)^{-1/2}. \quad (14.23)$$

The sign of $\omega_0''\omega_2$ is the same as the sign of σ ; the modulation equations are hyperbolic for $\sigma > 0$ and elliptic for $\sigma < 0$. For near-linear waves, the Sine-Gordon equation has $\sigma < 0$, so that in all the problems governed by this equation the near-linear wavetrains are unstable.

We shall return to all these questions after the formulation of the modulation equations has been studied in detail and extended to the fully nonlinear case.

14.3 The Variational Approach to Modulation Theory

The complete modulation equations are obtained in a particularly compact and significant form from the variational approach started in Chapter 11. We first see how to implement it for nonlinear problems using the Klein-Gordon equation as a typical example. General procedures then become apparent and we include these in the justification of the method.

In the Klein-Gordon case the periodic wavetrain is described by (14.4)–(14.5) and involves the parameters ω , k , and A . We need to find the equations satisfied by these parameters for a slowly varying wavetrain. Equation 14.1 is the Euler equation for the variational principle

$$\delta \int \int \left\{ \frac{1}{2} \varphi_t^2 - \frac{1}{2} \varphi_x^2 - V(\varphi) \right\} dx dt = 0, \quad (14.24)$$

as is easily verified from (11.74). The elementary solution corresponding to the solution $\varphi = a \cos(\theta + \eta)$ used in linear problems is $\varphi = \Psi(\theta)$. [A phase shift η can be added to (14.5), but it drops out of the modulation equations.] We therefore calculate the Lagrangian and its average value for $\varphi = \Psi(\theta)$; this is done keeping ω , k , and A constant. We have

$$L = \frac{1}{2} (\omega^2 - k^2) \Psi_\theta^2 - V(\Psi),$$

and the average value over one period in θ is

$$\bar{L} = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{1}{2} (\omega^2 - k^2) \Psi_\theta^2 - V(\Psi) \right\} d\theta. \quad (14.25)$$

In principle, the function Ψ is known completely from (14.5). However, we can avoid the integrated form and use (14.4) instead to express \bar{L} as a function of ω, k, A . We note the successive steps

$$\begin{aligned} \bar{L} &= \frac{1}{2\pi} \int_0^{2\pi} (\omega^2 - k^2) \Psi_\theta^2 d\theta - A \\ &= \frac{1}{2\pi} (\omega^2 - k^2) \int_0^{2\pi} \Psi_\theta d\Psi - A \\ &= \frac{1}{2\pi} \{2(\omega^2 - k^2)\}^{1/2} \oint \{A - V(\Psi)\}^{1/2} d\Psi - A. \end{aligned} \quad (14.26)$$

The final loop integral is a well-defined function of A , in which Ψ is now merely a dummy variable of integration. The notation $\mathcal{L}(\omega, k, A)$ is reserved for the final form of \bar{L} .

When ω, k, A are allowed to be slowly varying functions of x and t , we propose the average variational principle,

$$\delta \iint \mathcal{L}(\omega, k, A) dx dt = 0, \quad (14.27)$$

as before. This is viewed as a variational principle for $\theta(x, t)$ and $A(x, t)$, with $\omega = -\theta_t$, $k = \theta_x$; the variational equations are

$$\delta A: \quad \mathcal{L}_A = 0, \quad (14.28)$$

$$\delta \theta: \quad \frac{\partial}{\partial t} \mathcal{L}_\omega - \frac{\partial}{\partial x} \mathcal{L}_k = 0. \quad (14.29)$$

After the variations have been taken, we again work with ω, k, A , and add the consistency relation

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \quad (14.30)$$

obtained by eliminating θ . The equations and their derivation from (14.27) are, of course, the same as in the linear case with the minor change of amplitude variable from a to A . The only new ingredient in the nonlinear theory is the calculation of $\mathcal{L}(\omega, k, A)$.

Equation 14.28 is a functional relation between ω, k, A , which can only be the dispersion relation. For the Klein-Gordon example with \mathcal{L} given by (14.26), we confirm that it does indeed give the correct result (14.7). The system (14.28)–(14.30) is the exact nonlinear form for the modulation equations tentatively proposed in the approximation (14.18)–(14.19). Before discussing the properties of these equations and their various extensions we turn now to the question of how the variational approach may be justified.

14.4 Justification of the Variational Approach

It will be sufficient to consider in detail the case of one dimensional waves described by a variational principle

$$\delta \int \int L(\varphi_t, \varphi_x, \varphi) dx dt = 0. \quad (14.31)$$

The cases of more dimensions, more dependent variables and nonuniform media can all be treated similarly. The Euler equation for (14.31) is

$$\frac{\partial}{\partial t} L_1 + \frac{\partial}{\partial x} L_2 - L_3 = 0, \quad (14.32)$$

where the L_j denote the derivatives

$$L_1 = \frac{\partial L}{\partial \varphi_t}, \quad L_2 = \frac{\partial L}{\partial \varphi_x}, \quad L_3 = \frac{\partial L}{\partial \varphi}. \quad (14.33)$$

Equation 14.32 is a second order partial differential equation for $\varphi(x, t)$ and we assume that this has periodic wavetrain solutions of the appropriate type.

For problems of slow modulations a parameter ϵ will be introduced by the initial or boundary conditions (as discussed in various cases in Section 11.8); ϵ measures the ratio of a typical wavelength or period relative to a typical length or time scale of the modulation. We shall eventually suppose ϵ to be small, but we make no restriction on the magnitude of the amplitude, only that its variations are slow.

The first step is to describe a modulated wavetrain precisely. If x and t are measured on the scale of the typical wavelength and period, the slowly varying quantities are functions of $\epsilon x, \epsilon t$; modulation parameters such as k and ω should be functions of this type. Yet φ itself varies due to the relatively fast oscillations as well. To incorporate these requirements, φ is written explicitly as a function of a phase function θ and of $\epsilon x, \epsilon t$. Then θ is chosen as $\epsilon^{-1}\Theta(\epsilon x, \epsilon t)$ to provide the relatively fast oscillation and to give

the correct dependence of $k = \theta_x$ and $\omega = -\theta_t$ on $\epsilon x, \epsilon t$. We therefore take

$$\varphi = \Phi(\theta, X, T; \epsilon), \quad (14.34)$$

$$\theta = \epsilon^{-1}\Theta(X, T), \quad X = \epsilon x, T = \epsilon t. \quad (14.35)$$

We define

$$\nu(X, T) = -\omega(X, T) = \Theta_T, \quad k(X, T) = \Theta_X \quad (14.36)$$

as the negative frequency and wave number. (In this general discussion we work with $\nu = -\omega$ to preserve the symmetry between x and t .) The scaling has been arranged so that

$$\frac{\partial \varphi}{\partial t} = \nu \frac{\partial \Phi}{\partial \theta} + \epsilon \frac{\partial \Phi}{\partial T}, \quad \frac{\partial \varphi}{\partial x} = k \frac{\partial \Phi}{\partial \theta} + \frac{\partial \Phi}{\partial X};$$

the variations due to the oscillations and to the slow modulations appear separately.

In vibration problems for ordinary mechanical systems, x is absent and the method amounts to distinguishing two time scales explicitly. It has become known as "two-timing," which is a colorful and convenient name even when "double-crossing" x variations are also involved. The art of two-timing lies in the fact that although one starts and ends with the correct number of independent variables, the expanded form can be used to advantage at intermediate steps. In the present case φ is ultimately a function of x and t through (14.35), but in appropriate parts of the analysis Φ is treated as a function of the *three* variables θ, X, T independently. In the usual two-timing procedures, the extra flexibility allows the suppression of secular and other terms. Its use in conjunction with variational principles will be different but equivalent.

The geometrical optics (*WKBJ*) type of expansion discussed in Section 11.8 is equivalent to choosing

$$\varphi(x, t) \sim e^{i\epsilon^{-1}\Theta(\epsilon x, \epsilon t)} \sum \epsilon^n A_n(\epsilon x, \epsilon t) \quad (14.37)$$

from the outset. The two-timing version would work with

$$\Phi(\theta, X, T; \epsilon) \sim e^{i\theta} \sum \epsilon^n A_n(X, T) \quad (14.38)$$

to the same ultimate ends. In either case, the exponential dependence on θ is limited to linear problems. For nonlinear problems, the counterpart would be to take an expansion

$$\Phi(\theta, X, T; \epsilon) \sim \sum \epsilon^n \Phi^{(n)}(\theta, X, T), \quad (14.39)$$

and determine the functions $\Phi^{(n)}$ successively. However, in the equivalent variational approach we make no such initial expansion; we work with (14.34)–(14.36) directly and avoid much of the tedious manipulation of the more standard perturbation procedures.

When (14.34) and (14.35) are substituted into the basic Euler equation (14.32) we have

$$\nu \frac{\partial L_1}{\partial \theta} + \epsilon \frac{\partial L_1}{\partial T} + k \frac{\partial L_2}{\partial \theta} + \epsilon \frac{\partial L_2}{\partial X} - L_3 = 0, \quad (14.40)$$

where the arguments of the L_j are given by

$$L_j = L_j(\nu \Phi_\theta + \epsilon \Phi_T, k \Phi_\theta + \epsilon \Phi_X, \Phi). \quad (14.41)$$

The relation $\theta = \epsilon^{-1} \Theta(X, T)$ was used to obtain (14.40), but it is now dropped. This is the crucial step in two-timing. Equation 14.40 is now considered as an equation for the function $\Phi(\theta, X, T)$ of three *independent* variables θ, X, T . The equation also involves the function $\Theta(X, T)$ through its derivatives $\nu = \Theta_T$, $k = \Theta_X$; the original relations of Θ, ν and k to the argument θ in Φ are also dropped. It is clear that if satisfactory solutions for $\Phi(\theta, X, T)$ and $\Theta(X, T)$ can be found, then $\Phi(\epsilon^{-1} \Theta, X, T)$ solves the original problem. The extra flexibility in the choice of $\Theta(X, T)$ is used to assure satisfactory behavior of $\Phi(\theta, X, T)$.

The choice of $\Theta(X, T)$ will appear in different ways depending on the particular variant of the method, but they are equivalent. Here we shall impose from the outset the requirement that Φ and its derivatives be periodic in θ . [Other variants leave $\Theta(X, T)$ open at first, find unwanted secular terms proportional to θ in the general expression for Φ , and eliminate them by the choice of Θ .] The period may be normalized to 2π , so we impose the condition that Φ and its derivatives be 2π -periodic in θ . To implement this condition, we note that (14.40) may be written in conservation form as

$$\frac{\partial}{\partial \theta} \{ (\nu L_1 + k L_2) \Phi_\theta - L \} + \epsilon \frac{\partial}{\partial T} (\Phi_\theta L_1) + \epsilon \frac{\partial}{\partial X} (\Phi_\theta L_2) = 0. \quad (14.42)$$

Then, on integration from $\theta=0$ to 2π , the contributions of the first term cancel, from the periodicity requirement, and we have

$$\frac{\partial}{\partial T} \frac{1}{2\pi} \int_0^{2\pi} \Phi_\theta L_1 d\theta + \frac{\partial}{\partial X} \frac{1}{2\pi} \int_0^{2\pi} \Phi_\theta L_2 d\theta = 0. \quad (14.43)$$

Equations 14.40 and 14.43 are the two equations for $\Phi(\theta, X, T)$ and $\Theta(X, T)$.

It is a remarkable and surprising fact that these equations for Φ and Θ are just the variational equations for the variational principle

$$\delta \int \int \frac{1}{2\pi} \int_0^{2\pi} L(\nu\Phi_\theta + \epsilon\Phi_T, k\Phi_\theta + \epsilon\Phi_X, \Phi) d\theta dX dT = 0. \quad (14.44)$$

Variations $\delta\Phi$ lead to

$$\frac{\partial}{\partial\theta} L_{\Phi_\theta} + \frac{\partial}{\partial T} L_{\Phi_T} + \frac{\partial}{\partial X} L_{\Phi_X} - L_\Phi = 0$$

in the usual way, and with the particular form of L in (14.44) this is seen to be (14.40). Variations $\delta\Theta$ give

$$\frac{\partial}{\partial T} \bar{L}_\nu + \frac{\partial}{\partial X} \bar{L}_k = 0, \quad (14.45)$$

where

$$\bar{L} = \frac{1}{2\pi} \int_0^{2\pi} L(\nu\Phi_\theta + \epsilon\Phi_T, k\Phi_\theta + \epsilon\Phi_X, \Phi) d\theta; \quad (14.46)$$

this is (14.43). But, most striking of all, (14.44) is just an *exact* form of the average variational principle! Not only do we justify the variational approach, we obtain a powerful and compact basis for the entire perturbation analysis. Strangely enough, we have made no explicit assumption so far that ϵ is small. It is implicit, however, in the choice of the functional form of Φ and the requirement that Φ be periodic in θ .

In the lowest order approximation to (14.44), we have

$$\delta \int \int \bar{L}^{(0)} dX dT = 0, \quad (14.47)$$

$$\bar{L}^{(0)} = \frac{1}{2\pi} \int_0^{2\pi} L\{\nu\Phi_\theta^{(0)}, k\Phi_\theta^{(0)}, \Phi^{(0)}\} d\theta. \quad (14.48)$$

The variational equations are

$$\delta\Phi^{(0)}: \quad \frac{\partial}{\partial\theta} \{\nu L_1^{(0)} + k L_2^{(0)}\} - L_3^{(0)} = 0, \quad (14.49)$$

$$\delta\Theta: \quad \frac{\partial}{\partial T} \bar{L}_\nu^{(0)} + \frac{\partial}{\partial X} \bar{L}_k^{(0)} = 0; \quad (14.50)$$

these are the lowest order approximations to (14.40) and (14.45), of course. Since X, T derivatives of $\Phi^{(0)}$ do not occur in (14.49), it is effectively an

ordinary differential equation for $\Phi^{(0)}$ as a function of θ . An immediate first integral [the corresponding approximation to (14.42)] is

$$\{ \nu L_1^{(0)} + L_2^{(0)} \} \Phi_\theta^{(0)} - L^{(0)} = A(X, T). \quad (14.51)$$

Equations 14.49 and 14.51 are just the ordinary differential equations describing the uniform periodic wavetrain, but with the difference that the parameters ν, k, A are now functions of X, T . The dependence on θ is exactly the same as in the periodic wavetrain; the dependence of ν, k, A on X, T provides the modulation. The explicit separation of θ from X, T automatically allows integrations with respect to θ in which ν, k, A are held fixed; integrations such as those in (14.25) and (14.26) are now seen to be in this sense.

When the solution of (14.51) is combined with (14.47)–(14.48), we have exactly the variational approach proposed earlier. It is now justified as the first approximation in a formal perturbation scheme.

In the actual use of the method there is an important question of technique to be explained in general terms. As it stands (14.51) can be used to determine both the function $\Phi^{(0)}$ and the dispersion relation between ν, k, A . [See (14.5) and (14.7) for the Klein-Gordon example.] The manipulations in (14.26) show that by limited use of (14.51) in (14.48) the explicit determination of $\Phi^{(0)}$ (which is just Ψ changed to the expanded notation) can be avoided and the dispersion relation can be incorporated as an additional variational equation derivable from (14.47). This is much to be preferred. For then the form of the average Lagrangian is simplified and, more importantly, *all* the equations relating the slowly varying parameters ν, k, A are collected in the variational principle. The question is how to describe this procedure in general terms. The problem is how to extract from (14.51) enough information on the functional form of $\Phi^{(0)}$ and not use complete information on the dispersion relation. We now show how this may be done.

14.5 Optimal Use of the Variational Principle

In the linear case there is no difficulty in separating the functional form of $\Phi^{(0)}$ from the dispersion relation. We know in advance that the solution of (14.49) or (14.51) will take the form

$$\Phi^{(0)} = a \cos(\theta + \eta),$$

where $a(X, T)$ is the amplitude related to $A(X, T)$ and used instead of it.

The phase parameter $\eta(X, T)$ will drop out in forming the average Lagrangian (14.48) and plays no role in this lowest order approximation. This rather trivial information on $\Phi^{(0)}$ is the only information extracted from (14.51) and does not include the dispersion relation. When this $\Phi^{(0)}$ is substituted in (14.48) we have the function

$$\mathcal{L}(\nu, k, a) = \frac{1}{2\pi} \int_0^{2\pi} L(-\nu a \sin \theta, -ka \sin \theta, a \cos \theta) d\theta$$

for the average Lagrangian.

In the near-linear case there is also no difficulty. We may use the Stokes expansion

$$\Phi^{(0)} = a \cos(\theta + \eta) + a_2 \cos(\theta + \eta_2) + a_3 \cos(\theta + \eta_3) + \dots$$

as the required form of $\Phi^{(0)}$ without including the dispersion relation. The relations of $a_2, a_3, \dots, \eta_2, \eta_3, \dots$, to a and η may be taken from (14.49) or (14.51), or they may be left arbitrary and also determined from the variational principle. For example, in the Klein-Gordon problem with $V(\varphi)$ given by (14.10), we take

$$\Phi^{(0)} = a \cos \theta + a_3 \cos 3\theta + a_5 \cos 5\theta + \dots$$

(It is easy to see in advance that the odd cosine terms are sufficient.) Then*

$$\begin{aligned} \bar{L}^{(0)} &= \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{1}{2} (\nu^2 - k^2) \Phi_\theta^{(0)} - \frac{1}{2} \Phi^{(0)2} - \sigma \Phi^{(0)4} \right\} d\theta \\ &= \frac{1}{4} (\nu^2 - k^2 - 1) a^2 - \frac{3\sigma a^4}{8} + \left(2a_3^2 - \frac{1}{2} \sigma a^3 a_3 \right) + \dots \end{aligned}$$

Variation with respect to a_3 shows that $a_3 = \frac{1}{8} \sigma a^3$ in agreement with (14.11). On resubstitution of this expression for a_3 in $\bar{L}^{(0)}$ we have

$$\mathcal{L}(\nu, k, a) = \frac{1}{4} (\nu^2 - k^2 - 1) a^2 - \frac{3}{8} \sigma a^4 - \frac{1}{32} \sigma^2 a^6 + \dots \quad (14.52)$$

Variation of a now gives the dispersion relation (14.12).

In the fully nonlinear case it is harder to disentangle the functional form of $\Phi^{(0)}$ from the dispersion relation. However, it can be done by use of a Hamiltonian version of the equations.

*A term proportional to $(\nu^2 - k^2 - 1)a_3^2$ is omitted because the subsequent equations show that $(\nu^2 - k^2 - 1) = 0(a^2)$.

Hamiltonian Transformation.

The transformation will be applied here only to the lowest order approximation in (14.47)–(14.51), so to ease the notation we drop the superscript zero on all quantities. The idea is to eliminate the quantity Φ_θ in favor of $\partial L/\partial \Phi_\theta$ just as \dot{q} is eliminated in favor of a generalized momentum $p = \partial L/\partial \dot{q}$ in ordinary mechanics. A new variable is defined by

$$\Pi = \frac{\partial L}{\partial \Phi_\theta} = \nu L_1 + k L_2, \quad (14.53)$$

and the Hamiltonian $H(\Pi, \Phi; \nu, k)$ is defined by

$$H = \Phi_\theta \frac{\partial L}{\partial \Phi_\theta} - L = \Phi_\theta (\nu L_1 + k L_2) - L. \quad (14.54)$$

From the transformation alone we have

$$\frac{\partial \Phi}{\partial \theta} = \frac{\partial H}{\partial \Pi}, \quad (14.55)$$

and (14.49) provides

$$\frac{\partial \Pi}{\partial \theta} = - \frac{\partial H}{\partial \Phi}. \quad (14.56)$$

These replace the second order equation (14.49) for Φ by two first order equations for Φ and Π . The variational principle (14.47) may now be written with

$$\bar{L} = \frac{1}{2\pi} \int_0^{2\pi} (\Pi \Phi_\theta - H) d\theta. \quad (14.57)$$

Moreover, there is an important extension. In the original form the variation $\delta \Phi_\theta$ is tied to $\delta \Phi$; hence $\delta \Pi$ is tied to the variation of Φ through (14.53), and (14.55) is a consequence of the transformation not a variational equation. However, we simply observe that both (14.55) and (14.56) follow from (14.57) if Φ and Π are allowed to vary *independently*. We are therefore free to make this extension. The next thing to note is that (14.51) is just the energy integral

$$H(\Pi, \Phi; \nu, k) = A(X, T) \quad (14.58)$$

for (14.55) and (14.56). Moreover, in this form it provides only the function

$$\Pi(\Phi; \nu, k, A).$$

Without using the relation of Π to Φ_θ , which has now been turned into one of the variational equations, there is no way to deduce the dispersion relation as well. This achieves the required separation of (14.51) into information about the form of solutions (now provided by the dependence of Π on Φ) and the dispersion relation. Finally, since the stationary values of (14.57) are known to satisfy (14.58), we may restrict the variations to functions which already satisfy (14.58). Then (14.57) may be evaluated as

$$\mathcal{L}(\nu, k, A) = \frac{1}{2\pi} \oint \Pi(\Phi; \nu, k, A) d\Phi - A, \quad (14.59)$$

and $\Pi(\Phi; \nu, k, A)$ is the function determined from (14.58). The variational principle becomes

$$\delta \int \int \mathcal{L}(\nu, k, A) dX dT = 0.$$

The variation with respect to A is the only remnant of the variations of Φ and Π . The variational equations are now

$$\delta A: \quad \mathcal{L}_A = 0,$$

$$\delta \Theta: \quad \frac{\partial}{\partial T} \mathcal{L}_\nu + \frac{\partial}{\partial X} \mathcal{L}_k = 0,$$

and the consistency relation

$$\frac{\partial k}{\partial T} - \frac{\partial \nu}{\partial X} = 0,$$

is added. These are the equations (14.28)–(14.30) with $\nu = -\omega$.

In the Klein-Gordon example,

$$L = \frac{1}{2} (\nu^2 - k^2) \Phi_\theta^2 - V(\Phi),$$

the Hamiltonian transformation is

$$\Pi = \frac{\partial L}{\partial \Phi_\theta} = (\nu^2 - k^2) \Phi_\theta,$$

$$H = \Phi_\theta \frac{\partial L}{\partial \Phi_\theta} - L = \frac{1}{2} (\nu^2 - k^2)^{-1} \Pi^2 + V(\Phi).$$

The integral $H = A$ is solved as

$$\Pi = \{2(\nu^2 - k^2)\}^{1/2} \{A - V(\Phi)\}^{1/2},$$

and

$$\begin{aligned} \mathcal{L} &= \frac{1}{2\pi} \oint \Pi d\Phi - A \\ &= \frac{1}{2\pi} \{2(\nu^2 - k^2)\}^{1/2} \oint \{A - V(\Phi)\}^{1/2} d\Phi - A, \end{aligned}$$

in agreement with (14.26).

Naturally the Hamiltonian transformation can be used also in the linear or near-linear cases. The expressions for \mathcal{L} may then differ in form from those obtained previously, but of course the resulting variational equations are equivalent.

14.6 Comments on the Perturbation Scheme

The usual procedure in applying perturbation methods is to substitute suitable expansions in powers of ϵ directly into the differential equations of the problem, obtain a hierarchy of equations for the successive orders, and then take steps to ensure uniform validity. It was in this manner that the *results* of the variational approach were first verified by Luke (1966). The expansion (14.39) is substituted in the equation for φ to give equations that we may write schematically as

$$E_0\{\Phi^{(0)}\} = 0, \quad E_1\{\Phi^{(1)}, \Phi^{(0)}\} = F_1\{\Phi^{(0)}\}, \quad \text{and so on.}$$

The zeroth equation for $\Phi^{(0)}$ is equivalent to (14.49). It is solved for $\Phi^{(0)}$; the dispersion relation is obtained between ν, k, A , but their dependence on X, T is undetermined at this level. The equation for $\Phi^{(1)}$ involves only θ derivatives of $\Phi^{(1)}$ and so is effectively an ordinary differential equation. Its solution has unbounded terms proportional to θ , unless conditions are imposed on $F_1\{\Phi^{(0)}\}$. These "secular" terms must be suppressed to ensure uniform validity of the expansion for large θ . The required condition on $F_1\{\Phi^{(0)}\}$ leads to the further equation for ν, k, A , which completes the lowest order solution. In the subsequent equations for the $\Phi^{(n)}$, there are further parameters and further secular conditions.

The prior requirement that Φ be periodic is equivalent to the suppression of secular terms. Therefore the successive approximations to the

periodicity condition (14.43) would appear as secular conditions in the more traditional procedure. We see the advantage of starting from (14.42) and (14.43) even if that procedure were to be followed. But, better still, since (14.42) and (14.43) correspond to the variational principle (14.44), the expansion can be substituted directly in (14.44) and the variational principle used to generate both the equations for $\Phi^{(n)}$ and the secular conditions. Thus the variational approach should not be considered as a separate method. It includes the usual expansion approach, for which it streamlines the details and allows general results to be formulated.

There are other advantages. The variational principle (14.44) has been established independently of any assumed form for the dependence on ϵ . Furthermore, Θ may also be allowed to depend on ϵ ; it was taken independent of ϵ only for simplicity in the initial presentation. We may use expansions in powers of ϵ for Φ or Θ or both, but we are also free to take other forms. For example, in the near-linear case, we may use expansions in powers of the amplitude, or, what amounts to the same thing, Fourier series for Φ . This will be the choice in the discussion of higher order approximations in Section 15.5.

14.7 Extensions to More Variables

The extension to more space dimensions is immediate. The plane periodic wave solutions have $\varphi = \Psi(\theta)$ where $\theta = \theta(\mathbf{x}, t)$ depends on a vector \mathbf{x} and the propagation is in the direction of the vector wave number $\mathbf{k} = \theta_{\mathbf{x}}$. The average Lagrangian becomes $\mathcal{L}(\omega, \mathbf{k}, A)$ and modulations in space (i.e., slowly curving phase surfaces) also become possible. The modulation equations are (11.80)–(11.82). The justification of the last section requires only the obvious changes of replacing x, X, k by x_i, X_i, k_i and performing the corresponding summations when necessary.

The case of a single higher order equation goes through similarly with only minor extensions. There will be higher order derivatives in (14.31) and in all the later steps, but the extensions are straightforward.

The case of more dependent variables requires detailed comment. First, for a linear system in a set of functions $\varphi^{(\alpha)}(\mathbf{x}, t)$, periodic wavetrains may be described by

$$\varphi^{(\alpha)} = a_{\alpha} \cos \theta + b_{\alpha} \sin \theta.$$

The average Lagrangian calculated from this is a function of the two sets a_{α}, b_{α} , as well as ω and \mathbf{k} . The corresponding variational principle

$$\delta \int \int \mathcal{L}(\theta, \theta_{\mathbf{x}}, a_{\alpha}, b_{\alpha}) d\mathbf{x} dt = 0 \quad (14.60)$$

leads to the variational equations

$$\begin{aligned} \mathcal{L}_{a_\alpha} &= 0, & \mathcal{L}_{b_\alpha} &= 0, \\ \frac{\partial}{\partial t} \mathcal{L}_\omega - \frac{\partial}{\partial x_j} \mathcal{L}_{k_j} &= 0 & (14.61) \\ \frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} &= 0, & \frac{\partial k_i}{\partial x_j} - \frac{\partial k_j}{\partial x_i} &= 0. \end{aligned}$$

The set of equations $\mathcal{L}_{a_\alpha} = \mathcal{L}_{b_\alpha} = 0$ are linear and homogeneous (since \mathcal{L} is quadratic in a_α, b_α) and they may in general be solved to express the a_α and b_α in terms of single amplitude a . These expressions may be reinserted into the Lagrangian to give \mathcal{L} as a function $\mathcal{L}_1(\omega, \mathbf{k}, a)$, and the modulation equations are the same as in the single variable case. The substitution is permissible, since the restricted choice for a_α and b_α does satisfy the stationary conditions. The equivalence may also be verified directly, since

$$\begin{aligned} \mathcal{L}_{1a} &= \frac{\partial a_\alpha}{\partial a} \mathcal{L}_{a_\alpha} + \frac{\partial b_\alpha}{\partial a} \mathcal{L}_{b_\alpha} = 0, \\ \mathcal{L}_{1\omega} &= \mathcal{L}_\omega + \frac{\partial a_\alpha}{\partial \omega} \mathcal{L}_{a_\alpha} + \frac{\partial b_\alpha}{\partial \omega} \mathcal{L}_{b_\alpha} = \mathcal{L}_\omega, \end{aligned}$$

and similarly $\mathcal{L}_{1k_j} = \mathcal{L}_{k_j}$. In the course of the substitutions different expressions for \mathcal{L}_1 may result, depending on which relations are chosen, but the final equations are the same. The justification via two-timing proceeds as before.

For nonlinear problems, the usual situation concerns a system of equations with a corresponding Lagrangian $L\{\varphi_t^{(\alpha)}, \varphi_x^{(\alpha)}, \varphi^{(\alpha)}\}$ involving only the $\varphi^{(\alpha)}$ and their first derivatives. However, it is typical that for some of the φ , only the derivatives appear in L ; they are "potentials" in that only the derivatives φ_t, φ_x represent physical quantities. This requires a highly nontrivial extension with important mathematical and physical consequences. In the uniform wavetrain solution any potential variable $\tilde{\varphi}$ must be expressed as

$$\tilde{\varphi} = \boldsymbol{\beta} \cdot \mathbf{x} - \gamma t + \tilde{\Phi}(\boldsymbol{\theta}), \quad \boldsymbol{\theta} = \mathbf{k} \cdot \mathbf{x} - \omega t, \quad (14.62)$$

in order to ensure complete generality. The physical quantities involve only

$$\tilde{\varphi}_t = -\gamma - \omega \tilde{\Phi}_\theta, \quad \tilde{\varphi}_x = \boldsymbol{\beta} + \mathbf{k} \tilde{\Phi}_\theta, \quad (14.63)$$

and $-\gamma, \beta$ represent the mean values. These are important physical quantities; in water waves they give the mean fluid velocity and mean height, for example. Moreover, a most important nonlinear effect is the coupling of modulations in the wavetrain to similar slow variations in these mean quantities. Thus in the modulation theory the term $\beta \cdot \mathbf{x} - \gamma t$ must be generalized to a function $\tilde{\theta}(\mathbf{x}, t)$ and γ, β defined by

$$\gamma = -\tilde{\theta}_t, \quad \beta = \tilde{\theta}_x. \quad (14.64)$$

The function $\tilde{\theta}$ is similar to θ and is a pseudo-phase appearing in the problem. The quantities γ and β are a pseudo-frequency and a pseudo-wave number. Furthermore, each potential $\tilde{\varphi}$ has the term $L_{\tilde{\varphi}}$ missing in its Euler equation

$$\frac{\partial}{\partial t} L_{\tilde{\varphi}_t} + \frac{\partial}{\partial \mathbf{x}} \cdot L_{\tilde{\varphi}_x} = 0; \quad (14.65)$$

in the course of the analysis, this always allows an extra integral and an extra parameter B to be introduced similar to A . The triads (γ, β, B) are similar, although subsidiary, to the main triad (ω, \mathbf{k}, A) .

The two-timed form corresponding to (14.62) is

$$\tilde{\varphi}(\mathbf{x}, t) = \epsilon^{-1} \tilde{\Theta}(\mathbf{X}, T) + \tilde{\Phi}(\theta, \mathbf{X}, T; \epsilon),$$

where

$$\gamma(\mathbf{X}, T) = -\tilde{\Theta}_T, \quad \beta(\mathbf{X}, T) = \tilde{\Theta}_x, \quad \mathbf{X} = \epsilon \mathbf{x}, \quad T = \epsilon t,$$

and $\tilde{\Phi}$ is chosen to be periodic in θ . For a Lagrangian

$$L(\varphi_t, \varphi_x, \varphi, \tilde{\varphi}_t, \tilde{\varphi}_x),$$

it may be shown that the two-timed equations *and the conditions that Φ and $\tilde{\Phi}$ be 2π -periodic in θ* , are equivalent to an exact variational principle similar to (14.44). To lowest order it is

$$\delta \int \int \frac{1}{2\pi} \int_0^{2\pi} L(-\omega \Phi_\theta, \mathbf{k} \Phi_\theta, \Phi, -\gamma - \omega \tilde{\Phi}_\theta, \beta + \mathbf{k} \tilde{\Phi}_\theta) d\theta d\mathbf{X} dT = 0. \quad (14.66)$$

The variational equations corresponding to $\delta \Phi$ and $\delta \tilde{\Phi}$ determine the functions Φ and $\tilde{\Phi}$ and we have the two integrals

$$(-\omega L_1 + \mathbf{k} \cdot L_2 - \omega L_4 + \mathbf{k} \cdot L_5) \Phi_\theta - L = A(\mathbf{X}, T), \quad (14.67)$$

$$-\omega L_4 + \mathbf{k} \cdot L_5 = B(\mathbf{X}, T). \quad (14.68)$$

The variations $\delta\Theta$ and $\delta\tilde{\Theta}$ lead to the two secular conditions

$$\frac{\partial}{\partial t}\bar{L}_\omega - \frac{\partial}{\partial \mathbf{X}} \cdot \bar{L}_\mathbf{k} = 0, \quad \frac{\partial}{\partial T}\bar{L}_\gamma - \frac{\partial}{\partial \mathbf{X}} \cdot \bar{L}_\beta = 0.$$

Finally, a Hamiltonian transformation can be introduced as before, based on generalized momenta $\partial L/\partial\Phi_\theta$, $\partial L/\partial\tilde{\Phi}_\theta$, and (14.67)–(14.68) can be used to eliminate explicit dependence on Φ and $\tilde{\Phi}$ in favor of including the parameters A and B in the variational principle. We then have

$$\delta \int \mathcal{L}(\omega, \mathbf{k}, A, \gamma, \beta, B) d\mathbf{X} dT = 0, \quad (14.69)$$

and the variational equations are

$$\mathcal{L}_A = 0, \quad \mathcal{L}_B = 0, \quad (14.70)$$

$$\frac{\partial}{\partial T}\mathcal{L}_\omega - \frac{\partial}{\partial X_j}\mathcal{L}_{k_j} = 0, \quad \frac{\partial}{\partial T}\mathcal{L}_\gamma - \frac{\partial}{\partial X_j}\mathcal{L}_{\beta_j} = 0, \quad (14.71)$$

together with the consistency conditions

$$\frac{\partial k_i}{\partial T} - \frac{\partial \omega}{\partial X_i} = 0, \quad \frac{\partial \beta_i}{\partial T} - \frac{\partial \gamma}{\partial X_i} = 0, \quad (14.72)$$

$$\frac{\partial k_i}{\partial X_j} - \frac{\partial k_j}{\partial X_i} = 0, \quad \frac{\partial \beta_i}{\partial X_j} - \frac{\partial \beta_j}{\partial X_i} = 0. \quad (14.73)$$

Further details of the procedure and examples are given in the original papers (Whitham, 1965, 1967, 1970). An application to water waves on finite depth, where the extra parameters are crucially important, will be given in Chapter 16.

In this more general case, the energy equation that corresponds via Noether's theorem to the invariance of (14.69) with respect to shifts in T is now

$$\frac{\partial}{\partial T}(\omega\mathcal{L}_\omega + \gamma\mathcal{L}_\gamma - \mathcal{L}) + \frac{\partial}{\partial X_j}(-\omega\mathcal{L}_{k_j} - \gamma\mathcal{L}_{\beta_j}) = 0. \quad (14.74)$$

The momentum equation which corresponds to the invariance with respect to shifts in X_i is

$$\frac{\partial}{\partial T}(k_i\mathcal{L}_\omega + \beta_i\mathcal{L}_\gamma) + \frac{\partial}{\partial X_j}(-k_i\mathcal{L}_{k_j} - \beta_i\mathcal{L}_{\beta_j} + \mathcal{L}\delta_{ij}) = 0. \quad (14.75)$$

The final extension is to note that if the medium is not constant but

depends on \mathbf{X}, T , these will appear explicitly in the Lagrangian and therefore in \mathcal{L} . But the variational equations (14.70)–(14.73) are unchanged. The conservation equations (14.74) and (14.75), however, have terms $-\mathcal{L}_T, \mathcal{L}_{X_j}$, respectively, on the right hand sides [as may be verified directly from (14.70)–(14.73)].

14.8 Adiabatic Invariants

It was remarked previously that the quantities $\mathcal{L}_\omega, \mathcal{L}_{k_j}$ are analogous to the adiabatic invariants of classical mechanics. This correspondence can now be explored. In mechanics the setting is the theory of slow modulations for vibrating systems. The only independent variable is time, so in this case the modulations can be produced only by externally imposed changes in some parameter $\lambda(t)$. (This corresponds to a varying medium in the case of waves.) The classical theory is usually developed by Hamiltonian methods, which are not directly applicable to waves, but we may instead derive the simplest of the classical results by the methods developed here. For an oscillator with one degree of freedom $q(t)$ and one slowly varying parameter $\lambda(t)$, the variational principle is

$$\delta \int_{t_1}^{t_2} L(q, \dot{q}, \lambda) dt = 0,$$

and the variational equation is

$$\frac{d}{dt} L_{\dot{q}} - L_q = 0. \quad (14.76)$$

This case is covered by the arguments of Sections 14.3 and 14.4 simply by dropping the dependence on x . But it is useful to note the steps separately in the usual notation of mechanics. We follow the simple intuitive approach of Section 14.3; it is justified by Section 14.4.

We first calculate the average Lagrangian for the periodic motion with λ held fixed. If the period is $\tau = 2\pi/\nu$, then

$$\mathcal{L} = \frac{\nu}{2\pi} \int_0^\tau L dt. \quad (14.77)$$

In the periodic motion ($\lambda = \text{constant}$), (14.76) has the energy integral

$$\dot{q} L_{\dot{q}} - L = E. \quad (14.78)$$

This may be solved, in principle, to express \dot{q} as a function of (q, E, λ) and then the generalized momentum $p = L_{\dot{q}}$ can also be expressed as

$$p = p(q, E, \lambda).$$

If (14.78) is used to replace L in (14.77), we have

$$\begin{aligned} \mathcal{L} &= \frac{\nu}{2\pi} \int_0^\tau p \dot{q} dt - E \\ &= \frac{\nu}{2\pi} \oint p(q, E, \lambda) dq - E, \end{aligned} \quad (14.79)$$

where $\oint p dq$ means the integral over one complete period of oscillation [a closed loop in the (p, q) plane]. We now allow slow variations of λ , with consequent slow changes of ν and E , and use the average variational principle

$$\delta \int_{t_0}^{t_1} \mathcal{L}(\nu, E, \lambda) dt = 0. \quad (14.80)$$

It is again crucial to define ν as the derivative $\dot{\theta}$ of a phase $\theta(t)$ which increases by a constant normalized amount in one oscillation. This step looks perhaps less natural than in the waves case, but it becomes clear in the two-timing. The variations of (14.80) with respect to E and θ give

$$\mathcal{L}_E = 0, \quad \frac{d}{dt} \mathcal{L}_\nu = 0, \quad (14.81)$$

respectively. The first of these corresponds to the dispersion relation (14.28) and the second corresponds to the conservation equation (14.29). In view of (14.79) we have

$$\mathcal{L}_\nu = \frac{1}{2\pi} \oint p dq = \text{constant}, \quad (14.82)$$

which is just the classical result of the adiabatic invariant. As the system is modulated, ν and E vary individually but

$$I(\nu, E) = \frac{1}{2\pi} \oint p dq \quad (14.83)$$

remains constant. From (14.79) and (14.81) the period is given by

$$\tau = \frac{2\pi}{\nu} = I_E, \quad (14.84)$$

which is also classical. (An excellent account of the usual theory may be found in Landau and Lifshitz, 1960b, p. 154.)

In the two-timed form (14.59), the quantity Π is defined as $\partial L / \partial \Phi_\theta$, whereas the generalized momentum p is $\partial L / \partial \varphi_t$. Since $\varphi_t = \nu \Phi_\theta$, to lowest order, we have $\Pi = \nu p$ and the expressions (14.59) and (14.79) agree.

It is clear from this comparison that in the case of waves \mathcal{L}_ω is akin to the adiabatic invariant and that the \mathcal{L}_{k_j} are similar quantities for spatial modulations. In waves there is no need for an external drain of energy, since modulations in time can be balanced by modulations in space. If the medium is not constant, however, we have the additional effect of parameters analogous to λ , but the equation

$$\frac{\partial}{\partial t} \mathcal{L}_\omega - \frac{\partial}{\partial x_j} \mathcal{L}_{k_j} = 0 \quad (14.85)$$

still holds. The equation has become known as the conservation of wave action.

In the special case of a wavetrain uniform in space but responding to changes of the medium in time we have

$$\mathcal{L}_\omega = \text{constant.}$$

Alternatively, for a wavetrain of fixed frequency moving into a medium dependent on one space dimension x , we have

$$\mathcal{L}_k = \text{constant.}$$

These provide simple determinations of the amplitude. In general, modulations in space and time balance according to (14.85) and produce a *propagation* of the modulations.

The quantities \mathcal{L}_γ and \mathcal{L}_β in (14.71) are similar to \mathcal{L}_ω and \mathcal{L}_{k_j} . They arise because of the extra dependent variables, just as ordinary dynamical systems (involving only the time) may have further adiabatic invariants when there are more degrees of freedom. These wave systems have only one genuine frequency and so correspond to the degenerate cases of equal frequencies in dynamics.

14.9 Multiple-Phase Wavetrains

The general case of multiply periodic motions in dynamics would be mirrored in wave theory by wavetrains with more than one genuine phase function. It is straightforward to extend the formalism to this case but

questions of existence suggest caution. For a two-phase wavetrain, for example, the starting point would be a quasi-periodic solution

$$\varphi = \Psi(\theta_1, \theta_2), \quad \theta_1 = k_1 x - \omega_1 t, \quad \theta_2 = k_2 x - \omega_2 t, \quad (14.86)$$

in which Ψ is 2π periodic in both θ_1 and θ_2 . One would then go on to handle modulation theory as before. However, even in ordinary dynamics questions of the existence of quasi-periodic solutions are difficult ones in the nonlinear case, involving the well-known problems of small divisors, so there may be considerable difficulties hidden under the formalism. If the existence of solutions (14.86) and of neighboring modulated solutions is simply assumed, modulation equations can be developed as before. Ablowitz and Benney (1970) and Ablowitz (1971) have pursued some of the consequences. Delaney (1971) notes that the variational formalism goes through. If modulated wavetrains can be described by

$$\begin{aligned} \varphi &= \Phi(\theta_1, \theta_2, X, T; \epsilon), \\ \theta_1 &= \epsilon^{-1} \Theta_1(X, T), \quad \theta_2 = \epsilon^{-1} \Theta_2(X, T), \end{aligned}$$

it is straightforward to show that the two-timed equation for Φ and the two periodicity conditions follows from the variational principle

$$\delta \int \int \bar{L} dX dT = 0,$$

$$\bar{L} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} L(\nu_1 \Phi_{\theta_1} + \nu_2 \Phi_{\theta_2} + \epsilon \Phi_T, k_1 \Phi_{\theta_1} + k_2 \Phi_{\theta_2} + \epsilon \Phi_X, \Phi) d\theta_1 d\theta_2.$$

Modulation equations can then be developed as before.

14.10 Effects of Damping

As in Hamiltonian dynamics, the variational formalism applies naturally to conservative systems; dissipative effects have to be tacked on a little awkwardly as nonzero right hand sides to the previous equations. However, various canonical forms can be maintained and the left hand sides can still be written in terms of the Lagrangian. To illustrate this, we consider as a specific example the equation

$$\varphi_{tt} - \varphi_{xx} + V'(\varphi) = -\epsilon D(\varphi, \varphi_t),$$

where $\epsilon D(\varphi, \varphi_t)$ represents small dissipative effects. The two-timed equa-

tion corresponding to (14.42) is

$$\begin{aligned} & \frac{\partial}{\partial \theta} \left\{ \frac{1}{2} (\nu^2 - k^2) \Phi_\theta^2 + V(\Phi) - \frac{1}{2} \epsilon^2 \Phi_T^2 + \frac{1}{2} \epsilon^2 \Phi_X^2 \right\} \\ & + \epsilon \frac{\partial}{\partial T} \{ \nu \Phi_\theta^2 + \epsilon \Phi_\theta \Phi_T \} - \epsilon \frac{\partial}{\partial X} \{ k \Phi_\theta^2 + \epsilon \Phi_\theta \Phi_X \} \\ & = -\epsilon \Phi_\theta D(\Phi, \nu \Phi_\theta + \epsilon \Phi_T). \end{aligned}$$

To lowest order, we have

$$\frac{1}{2} (\nu^2 - k^2) \Phi_\theta^2 + V(\Phi) = A(X, T) \quad (14.87)$$

and the periodicity condition

$$\frac{\partial}{\partial T} \int_0^{2\pi} \nu \Phi_\theta^2 d\theta - \frac{\partial}{\partial X} \int_0^{2\pi} k \Phi_\theta^2 d\theta = - \int_0^{2\pi} \Phi_\theta D(\Phi, \nu \Phi_\theta) d\theta. \quad (14.88)$$

From (14.87), Φ_θ can be expressed as a function of Φ , ν , k , A , and the integrals in (14.88) can all be written as loop integrals. We have

$$\frac{\partial}{\partial T} \mathcal{L}_\nu + \frac{\partial}{\partial X} \mathcal{L}_k = -\mathcal{D}, \quad (14.89)$$

where

$$\mathcal{L}(\nu, k, A) = \frac{1}{2\pi} \{ 2(\nu^2 - k^2) \}^{1/2} \oint \{ A - V(\Phi) \}^{1/2} d\Phi - A,$$

as before, and

$$\mathcal{D}(\nu, k, A) = \frac{1}{2\pi} \oint D(\Phi, \Phi_\theta) d\Phi.$$

To (14.89) are added

$$\mathcal{L}_A = 0, \quad \frac{\partial k}{\partial T} - \frac{\partial \nu}{\partial X} = 0, \quad (14.90)$$

to complete the set of equations for ν , k , A . Equation 14.89 shows the loss in wave action due to dissipation.

Here we have returned to two-timing on the equations but retained the canonical forms suggested by the Lagrangian for the conservative part. This is obviously less desirable than two-timing directly some extended principle. Recently Jimenez (1972) has had some success in deriving results such as (14.89) by Prigogine's approach to irreversible systems (Donnelly et al., 1966).